# Chapter 1

# Expected Utility and Risk Aversion

Asset prices are determined by investors' risk preferences and by the distributions of assets' risky future payments. Economists refer to these two bases of prices as investor "tastes" and the economy's "technologies" for generating asset returns. A satisfactory theory of asset valuation must consider how individuals allocate their wealth among assets having different future payments. This chapter explores the development of expected utility theory, the standard approach for modeling investor choices over risky assets. We first analyze the conditions that an individual's preferences must satisfy to be consistent with an expected utility function. We then consider the link between utility and risk-aversion, and how risk-aversion leads to risk premia for particular assets. Our final topic examines how risk-aversion affects an individual's choice between a risky and a risk-free asset.

Modeling investor choices with expected utility functions is widely-used. However, significant empirical and experimental evidence has indicated that

individuals sometimes behave in ways inconsistent with standard forms of expected utility. These findings have motivated a search for improved models of investor preferences. Theoretical innovations both within and outside the expected utility paradigm are being developed, and examples of such advances are presented in later chapters of this book.

## 1.1 Preferences when Returns are Uncertain

Economists typically analyze the price of a good or service by modeling the nature of its supply and demand. A similar approach can be taken to price an asset. As a starting point, let us consider the modeling of an investor's demand for an asset. In contrast to a good or service, an asset does not provide a current consumption benefit to an individual. Rather, an asset is a vehicle for saving. It is a component of an investor's financial wealth representing a claim on *future* consumption or purchasing power. The main distinction between assets is the difference in their future payoffs. With the exception of assets that pay a risk-free return, assets' payoffs are random. Thus, a theory of the demand for assets needs to specify investors' preferences over different, uncertain payoffs. In other words, we need to model how investors choose between assets that have different probability distributions of returns. In this chapter we assume an environment where an individual chooses among assets that have random payoffs at a single future date. Later chapters will generalize the situation to consider an individual's choices over multiple periods among assets paying returns at multiple future dates.

Let us begin by considering potentially relevant criteria that individuals might use to rank their preferences for different risky assets. One possible measure of the attractiveness of an asset is the average or *expected value* of its payoff. Suppose an asset offers a single random payoff at a particular

future date, and this payoff has a discrete distribution with $n$ possible outcomes, $(x_1, ..., x_n)$, and corresponding probabilities $(p_1, ..., p_n)$, where $\sum_{i=1}^{n} p_i = 1$ and $p_i \geq 0$.[1] Then the expected value of the payoff (or, more simply, the expected payoff) is $\bar{x} \equiv E[\widetilde{x}] = \sum_{i=1}^{n} p_i x_i$.

Is it logical to think that individuals value risky assets based solely on the assets' expected payoffs? This valuation concept was the prevailing wisdom until 1713 when Nicholas Bernoulli pointed out a major weakness. He showed that an asset's expected payoff was unlikely to be the only criterion that individuals use for valuation. He did it by posing the following problem that became known as the "St. Petersberg Paradox:"

> Peter tosses a coin and continues to do so until it should land "heads" when it comes to the ground. He agrees to give Paul one ducat if he gets "heads" on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that on each additional throw the number of ducats he must pay is doubled.[2] Suppose we seek to determine Paul's expectation (of the payoff that he will receive).

Interpreting Paul's prize from this coin flipping game as the payoff of a risky asset, how much would he be willing to pay for this asset if he valued it based on its expected value? If the number of coin flips taken to first arrive at a heads is $i$, then $p_i = \left(\frac{1}{2}\right)^i$ and $x_i = 2^{i-1}$ so that the expected payoff equals

[1] As is the case in the following example, $n$, the number of possible outcomes, may be infinite.

[2] A ducat was a 3.5 gram gold coin used throughout Europe.

$$\begin{aligned} \bar{x} &= \sum_{i=1}^{\infty} p_i x_i = \tfrac{1}{2}1 + \tfrac{1}{4}2 + \tfrac{1}{8}4 + \tfrac{1}{16}8 + ... \\ &= \tfrac{1}{2}(1 + \tfrac{1}{2}2 + \tfrac{1}{4}4 + \tfrac{1}{8}8 + ... \\ &= \tfrac{1}{2}(1 + 1 + 1 + 1 + ... = \infty \end{aligned} \tag{1.1}$$

The "paradox" is that the expected value of this asset is infinite, but, intuitively, most individuals would pay only a moderate, not infinite, amount to play this game. In a paper published in 1738, Daniel Bernoulli, a cousin of Nicholas, provided an explanation for the St. Petersberg Paradox by introducing the concept of *expected utility.*[3] His insight was that an individual's utility or "felicity" from receiving a payoff could differ from the size of the payoff and that people cared about the expected utility of an asset's payoffs, not the expected value of its payoffs. Instead of valuing an asset as $\overline{x} = \sum_{i=1}^{n} p_i x_i$, its value, $V$, would be

$$V \equiv E\left[U\left(\widetilde{x}\right)\right] = \sum_{i=1}^{n} p_i U_i \tag{1.2}$$

where $U_i$ is the utility associated with payoff $x_i$. Moreover, he hypothesized that the "utility resulting from any small increase in wealth will be inversely proportionate to the quantity of goods previously possessed." In other words, the greater an individual's wealth, the smaller is the added (or marginal) utility received from an additional increase in wealth. In the St. Petersberg Paradox, prizes, $x_i$, go up at the same rate that the probabilities decline. To obtain a finite valuation, the trick is to allow the utility of prizes, $U_i$, to increase slower

[3] An English translation of Daniel Bernoulli's original Latin paper is printed in Econometrica (Bernoulli 1954). Another Swiss mathematician, Gabriel Cramer, offered a similar solution in 1728.

than the rate that probabilities decline. Hence, Daniel Bernoulli introduced the principle of a *diminishing marginal utility of wealth* (as expressed in his quote above) to resolve this paradox.

The first complete axiomatic development of expected utility is due to John von Neumann and Oskar Morgenstern (von Neumann and Morgenstern 1944). Von Neumann, a renowned physicist and mathematician, initiated the field of *game theory*, which analyzes strategic decision making. Morgenstern, an economist, recognized the field's economic applications and, together, they provided a rigorous basis for individual decision-making under uncertainty. We now outline one aspect of their work, namely, to provide conditions that an individual's preferences must satisfy for these preferences to be consistent with an expected utility function.

Define a *lottery* as an asset that has a risky payoff and consider an individual's optimal choice of a lottery (risky asset) from a given set of different lotteries. All lotteries have possible payoffs that are contained in the set $\{x_1, ..., x_n\}$. In general, the elements of this set can be viewed as different, uncertain outcomes. For example, they could be interpreted as particular consumption levels (bundles of consumption goods) that the individual obtains in different states of nature or, more simply, different monetary payments received in different states of the world. A given lottery can be characterized as an ordered set of probabilities $P = \{p_1, ..., p_n\}$, where, of course, $\sum_{i=1}^{n} p_i = 1$ and $p_i \geq 0$. A different lottery is characterized by another set of probabilities, for example, $P^* = \{p_1^*, ..., p_n^*\}$. Let $\succ$, $\prec$, and $\sim$ denote preference and indifference between lotteries.[4] We will show that if an individual's preferences satisfy the following conditions (axioms), then these preferences can be represented by a real-valued

[4] Specifically, if an individual prefers lottery $P$ to lottery $P^*$, this can be denoted as $P \succ P^*$ or $P^* \prec P$. When the individual is indifferent between the two lotteries, this is written as $P \sim P^*$. If an individual prefers lottery $P$ to lottery $P^*$or she is indifferent between lotteries $P$ and $P^*$, this is written as $P \succeq P^*$ or $P^* \preceq P$.

utility function defined over a given lottery's probabilities, that is, an expected utility function $V(p_1, ..., p_n)$.

Axioms:

1) *Completeness*

For any two lotteries $P^*$ and $P$, either $P^* \succ P$, or $P^* \prec P$, or $P^* \sim P$.

2) *Transitivity*

If $P^{**} \succeq P^*$and $P^* \succeq P$, then $P^{**} \succeq P$.

3) *Continuity*

If $P^{**} \succeq P^* \succeq P$, there exists some $\lambda \in [0, 1]$ such that $P^* \sim \lambda P^{**} + (1-\lambda)P$, where $\lambda P^{**} + (1 - \lambda)P$ denotes a "compound lottery," namely with probability $\lambda$ one receives the lottery $P^{**}$ and with probability $(1 - \lambda)$ one receives the lottery $P$.

These three axioms are analogous to those used to establish the existence of a real-valued utility function in standard consumer choice theory.[5] The fourth axiom is unique to expected utility theory and, as we later discuss, has important implications for the theory's predictions.

4) *Independence*

For any two lotteries $P$ and $P^*$, $P^* \succ P$ if for all $\lambda \in$ (0,1] and all $P^{**}$:

$$\lambda P^* + (1 - \lambda)P^{**} \succ \lambda P + (1 - \lambda)P^{**}$$

Moreover, for any two lotteries $P$ and $P^\dagger$, $P \sim P^\dagger$ if for all $\lambda \in$(0,1] and all $P^{**}$:

[5] A primary area of microeconomics analyzes a consumer's optimal choice of multiple goods (and services) based on their prices and the consumer's budget contraint. In that context, utility is a function of the quantities of multiple goods consumed. References on this topic include (Kreps 1990), (Mas-Colell, Whinston, and Green 1995), and (Varian 1992). In contrast, the analysis of this chapter expresses utility as a function of the individual's wealth. In future chapters, we introduce multi-period utility functions where utility becomes a function of the individual's overall consumption at multiple future dates. Financial economics typically bypasses the individual's problem of choosing among different consumption goods and focuses on how the individual chooses a total quantity of consumption at different points in time and different states of nature.

$$\lambda P + (1-\lambda)P^{**} \sim \lambda P^{\dagger} + (1-\lambda)P^{**}$$

To better understand the meaning of the independence axiom, note that $P^*$ is preferred to $P$ by assumption. Now the choice between $\lambda P^* + (1-\lambda)P^{**}$ and $\lambda P + (1-\lambda)P^{**}$ is equivalent to a toss of a coin that has a probability $(1-\lambda)$ of landing "tails", in which case both compound lotteries are equivalent to $P^{**}$, and a probability $\lambda$ of landing "heads," in which case the first compound lottery is equivalent to the single lottery $P^*$ and the second compound lottery is equivalent to the single lottery $P$. Thus, the choice between $\lambda P^* + (1-\lambda)P^{**}$ and $\lambda P + (1-\lambda)P^{**}$ is equivalent to being asked, prior to the coin toss, if one would prefer $P^*$ to $P$ in the event the coin lands "heads."

It would seem reasonable that should the coin land "heads," we would go ahead with our original preference in choosing $P^*$ over $P$. The independence axiom assumes that preferences over the two lotteries are independent of the way in which we obtain them.[6] For this reason, the independence axiom is also known as the "no regret" axiom. However, experimental evidence finds some systematic violations of this independence axiom, making it a questionable assumption for a theory of investor preferences. For example, the Allais Paradox is a well-known choice of lotteries that, when offered to individuals, leads most to violate the independence axiom.[7] Machina (Machina 1987) summarizes violations of the independence axiom and reviews alternative approaches to modeling risk preferences. In spite of these deficiencies, von Neumann - Morgenstern expected utility theory continues to be a useful and common ap-

[6] In the context of standard consumer choice theory, $\lambda$ would be interpreted as the amount (rather than probability) of a particular good or bundle of goods consumed (say $C$) and $(1-\lambda)$ as the amount of another good or bundle of goods consumed (say $C^{**}$). In this case, it would not be reasonable to assume that the choice of these different bundles is independent. This is due to some goods being substitutes or complements with other goods. Hence, the validity of the independence axiom is linked to outcomes being uncertain (risky), that is, the interpretation of $\lambda$ as a probability rather than a deterministic amount.

[7] A similar example is given as an exercise at the end of this chapter.

proach to modeling investor preferences, though research exploring alternative paradigms is growing.[8]

The final axiom is similar to the independence and completeness axioms.

5) *Dominance*

Let $P^1$ be the compound lottery $\lambda_1 P^{\ddagger} + (1-\lambda_1)P^{\dagger}$ and $P^2$ be the compound lottery $\lambda_2 P^{\ddagger} + (1-\lambda_2)P^{\dagger}$. If $P^{\ddagger} \succ P^{\dagger}$, then $P^1 \succ P^2$ if and only if $\lambda_1 > \lambda_2$.

Given preferences characterized by the above axioms, we now show that the choice between any two (or more) arbitrary lotteries is that which has the higher (highest) expected utility.

The completeness axiom's ordering on lotteries naturally induces an ordering on the set of outcomes. To see this, define an "elementary" or "primitive" lottery, $e_i$, which returns outcome $x_i$ with probability 1 and all other outcomes with probability zero, that is, $e_i = \{\mathrm{p}_1, ..., \mathrm{p}_{i-1}, \mathrm{p}_i, \mathrm{p}_{i+1}, ..., \mathrm{p}_n\} = \{0, ..., 0, 1, 0, ...0\}$ where $\mathrm{p}_i = 1$ and $\mathrm{p}_j = 0 \ \forall j \neq i$. Without loss of generality, suppose that the outcomes are ordered such that $e_n \succeq e_{n-1} \succeq ... \succeq e_1$. This follows from the completeness axiom for this case of $n$ elementary lotteries. Note that this ordering of the elementary lotteries may not necessarily coincide with a ranking of the elements of $x$ strictly by the size of their monetary payoffs, as the state of nature for which $x_i$ is the outcome may differ from the state of nature for which $x_j$ is the outcome, and these states of nature may have different effects on how an individual values the same monetary outcome. For example, $x_i$ may be received in a state of nature when the economy is depressed, and monetary payoffs may be highly valued in this state of nature. In contrast, $x_j$ may be received in a state of nature characterized by high economic expansion, and monetary payments may not be as highly valued. Therefore, it may be that $e_i \succ e_j$ even if the monetary payment corresponding to $x_i$ was less than that

[8] This research includes "behavioral finance," a field that encompasses alternatives to both expected utility theory and market efficiency. An example of how a behavioral finance - type utility specification can impact asset prices will be presented in Chapter 15.

corresponding to $x_j$.

From the continuity axiom, we know that for each $e_i$, there exists a $U_i \in [0,1]$ such that

$$e_i \sim U_i e_n + (1 - U_i)e_1 \tag{1.3}$$

and for $i = 1$, this implies $U_1 = 0$ and for $i = n$, this implies $U_n = 1$. The values of the $U_i$ weight the most and least preferred outcomes such that the individual is just indifferent between a combination of these polar payoffs and the payoff of $x_i$. The $U_i$ can adjust for both differences in monetary payoffs and differences in the states of nature during which the outcomes are received.

Now consider a given arbitrary lottery, $P = \{p_1, ..., p_n\}$. This can be considered a compound lottery over the $n$ elementary lotteries, where elementary lottery $e_i$ is obtained with probability $p_i$. By the independence axiom, and using equation (1.3), the individual is indifferent between the compound lottery, $P$, and the following lottery given on the right-hand-side of the equation below:

$$\begin{aligned} p_1e_1 + ... + p_ne_n \quad \sim \quad & p_1e_1 + ... + p_{i-1}e_{i-1} + p_i\left[U_ie_n + (1 - U_i)e_1\right] \\ & + p_{i+1}e_{i+1} + ... + p_ne_n \end{aligned} \tag{1.4}$$

where we have used the indifference relation in equation (1.3) to substitute for $e_i$ on the right hand side of (1.4). By repeating this substitution for all $i$, $i = 1, ..., n$, we see that the individual will be indifferent between $P$, given by the left hand side of (1.4), and

$$p_1e_1 + ... + p_ne_n \sim \left(\sum_{i=1}^{n} p_iU_i\right) e_n + \left(1 - \sum_{i=1}^{n} p_iU_i\right) e_1 \tag{1.5}$$

Now define $\Lambda \equiv \sum_{i=1}^{n} p_i U_i$. Thus, we see that lottery $P$ is equivalent to a compound lottery consisting of a $\Lambda$ probability of obtaining elementary lottery $e_n$ and a $(1-\Lambda)$ probability of obtaining elementary lottery $e_1$. In a similar manner, we can show that any other arbitrary lottery $P^* = \{p_1^*, ..., p_n^*\}$ is equivalent to a compound lottery consisting of a $\Lambda^*$ probability of obtaining $e_n$ and a $(1-\Lambda^*)$ probability of obtaining $e_1$, where $\Lambda^* \equiv \sum_{i=1}^{n} p_i^* U_i$.

Thus, we know from the dominance axiom that $P^* \succ P$ if and only if $\Lambda^* > \Lambda$, which implies $\sum_{i=1}^{n} p_i^* U_i > \sum_{i=1}^{n} p_i U_i$. So defining an expected utility function as

$$V(p_1, ..., p_n) = \sum_{i=1}^{n} p_i U_i \tag{1.6}$$

will imply that $P^* \succ P$ if and only if $V(p_1^*, ..., p_n^*) > V(p_1, ..., p_n)$.

The function given in equation (1.6) is known as von Neumann - Morgenstern expected utility. Note that it is linear in the probabilities and is unique up to a linear monotonic transformation.[9] This implies that the utility function has "cardinal" properties, meaning that it does not preserve preference orderings for all strictly increasing transformations.[10] For example, if $U_i = U(x_i)$, an individual's choice over lotteries will be the same under the transformation $aU(x_i) + b$, but not a non-linear transformation that changes the "shape" of $U(x_i)$.

The von Neumann-Morgenstern expected utility framework may only partially explain the phenomenon illustrated by the St. Petersberg Paradox. Suppose an individual's utility is given by the square root of a monetary payoff, that is, $U_i = U(x_i) = \sqrt{x_i}$. This is a monotonically increasing, concave function of

[9] The intuition for why expected utility is unique up to a linear transformation can be traced to equation (1.3). The derivation chose to compare elementary lottery $i$ in terms of the least and most preferred elementary lotteries. However, other bases for ranking a given lottery are possible.

[10] An "ordinal" utility function preserves preference orderings for *any* strictly increasing transformation, not just linear ones. The utility functions defined over multiple goods and used in standard consumer theory are ordinal measures.

$x$, which, here, is assumed to be simply a monetary amount (in units of ducats). Then the individual's expected utility of the St. Petersberg payoff is

$$\begin{aligned} V &= \sum_{i=1}^{n} p_i U_i = \sum_{i=1}^{\infty} \frac{1}{2^i} \sqrt{2^{i-1}} = \sum_{i=2}^{\infty} 2^{-\frac{i}{2}} \\ &= 2^{-\frac{2}{2}} + 2^{-\frac{3}{2}} + ... \\ &= \sum_{i=0}^{\infty} \left( \frac{1}{\sqrt{2}} \right)^i - 1 - \frac{1}{\sqrt{2}} = \frac{1}{2 - \sqrt{2}} \cong 1.707 \end{aligned} \tag{1.7}$$

which is finite. This individual would get the same expected utility from receiving a certain payment of $1.707^2 \cong 2.914$ ducats since $V = \sqrt{2.914}$also gives expected (and actual) utility of 1.707. Hence we can conclude that the St. Petersberg gamble would be worth 2.914 ducats to this square-root utility maximizer.

However, the reason that this is not a complete resolution of the paradox is that one can always construct a "super St. Petersberg paradox" where even expected utility is infinite. Note that in the regular St. Petersberg paradox, the probability of winning declines at rate $2^i$ while the winning payoff increases at rate $2^i$. In a super St. Petersberg paradox, we can make the winning payoff increase at a rate $x_i = U^{-1}(2^{i-1})$ and expected utility would no longer be finite. If we take the example of square-root utility, let the winning payoff be $x_i = 2^{2i-2}$, that is, $x_1 = 1$, $x_2 = 4$, $x_3 = 16$, etc. In this case, the expected utility of the super St. Petersberg payoff by a square-root expected utility maximizer is

$$V = \sum_{i=1}^{n} p_i U_i = \sum_{i=1}^{\infty} \frac{1}{2^i} \sqrt{2^{2i-2}} = \infty \tag{1.8}$$

Should we be concerned by the fact that if we let the prizes grow quickly enough, we can get infinite expected utility (and valuations) for any chosen form of expected utility function? Maybe not. One could argue that St. Petersberg

games are unrealistic, particularly ones where the payoffs are assumed to grow rapidly. The reason is that any person offering this asset has finite wealth (even Bill Gates). This would set an upper bound on the amount of prizes that could feasibly be paid, making expected utility, and even the expected value of the payoff, finite.

The von Neumann-Morgenstern expected utility approach can be generalized to the case of a continuum of outcomes and lotteries having continuous probability distributions. For example, if outcomes are a possibly infinite number of purely monetary payoffs or consumption levels denoted by the variable $x$, a subset of the real numbers, then a generalized version of equation (1.6) is

$$V(F) = E[U(\widetilde{x})] = \int U(x)\, dF(x) \tag{1.9}$$

where $F(x)$ is a given lottery's cumulative distribution function over the payoffs, $x$.[11] Hence, the generalized lottery represented by the distribution function $F$ is analogous to our previous lottery represented by the discrete probabilities $P = \{p_1, ..., p_n\}$.

Thus far, our discussion of expected utility theory has said little regarding an appropriate specification for the utility function, $U(x)$. We now turn to a discussion of how the form of this function affects individuals' risk preferences.

## 1.2 Risk Aversion and Risk Premia

As mentioned in the previous section, Daniel Bernoulli proposed that utility functions should display diminishing marginal utility, that is, $U(x)$ should be an increasing but concave function of wealth. He recognized that this concavity implies that an individual will be risk averse. By risk averse we mean that

[11] When the random payoff, $\widetilde{x}$, is absolutely continuous, then expected utility can be written in terms of the probability density function, $f(x)$, as $V(f) = \int U(x) f(x)\, dx$.

the individual would not accept a "fair" lottery (asset), where a fair or "pure risk" lottery is defined as one that has an expected value of zero. To see the relationship between fair lotteries and concave utility, consider the following example. Let there be a lottery that has a random payoff, $\widetilde{\varepsilon}$, where

$$\widetilde{\varepsilon} = \begin{cases} \varepsilon_1 \text{with probability } p \\ \varepsilon_2 \text{ with probability } 1-p \end{cases} \tag{1.10}$$

The requirement that it be a fair lottery restricts its expected value to equal zero:

$$E\left[\widetilde{\varepsilon}\right] = p\varepsilon_1 + (1-p)\varepsilon_2 = 0 \tag{1.11}$$

which implies $\varepsilon_1/\varepsilon_2 = -\left(1-p\right)/p$, or, solving for $p$, $p = -\varepsilon_2/\left(\varepsilon_1 - \varepsilon_2\right)$. Of course since $0 < p < 1$, $\varepsilon_1$ and $\varepsilon_2$ are of opposite signs.

Now suppose a von Neumann-Morgenstern expected utility maximizer whose current wealth equals $W$ is offered the above lottery. Would this individual accept it, that is, would she place a positive value on this lottery?

If the lottery is accepted, expected utility is given by $E\left[U\left(W+\widetilde{\varepsilon}\right)\right]$. Instead, if it is not accepted, expected utility is given by $E\left[U\left(W\right)\right] = U\left(W\right)$. Thus, an individual's refusal to accept a fair lottery implies

$$U\left(W\right) > E\left[U\left(W+\widetilde{\varepsilon}\right)\right] = pU\left(W+\varepsilon_1\right) + (1-p)U\left(W+\varepsilon_2\right) \tag{1.12}$$

To show that this is equivalent to having a concave utility function, note that $U\left(W\right)$ can be re-written as

$$U(W) = U\left(W + p\varepsilon_1 + (1-p)\varepsilon_2\right) \tag{1.13}$$

since $p\varepsilon_1 + (1-p)\varepsilon_2 = 0$ by the assumption that the lottery is fair. Re-writing inequality (1.12), we have

$$U\left(W + p\varepsilon_1 + (1-p)\varepsilon_2\right) > pU\left(W + \varepsilon_1\right) + (1-p)U\left(W + \varepsilon_2\right) \tag{1.14}$$

which is the definition of $U$ being a concave function. A function is concave if a line joining any two points of the function lies entirely below the function. When $U(W)$ is concave, a line connecting the points $U(W+\varepsilon_2)$ to $U(W+\varepsilon_1)$ lies below $U(W)$ for all $W$ such that $W+\varepsilon_2 < W < W+\varepsilon_1$. As shown in Figure 1.1, $pU(W+\varepsilon_1) + (1-p)U(W+\varepsilon_2)$ is exactly the point on this line directly below $U(W)$. This is clear by substituting $p = -\varepsilon_2/(\varepsilon_1 - \varepsilon_2)$. Note that when $U(W)$ is a continuous, second differentiable function, concavity implies that its second derivative, $U''(W)$, is less than zero.

To show the reverse, that concavity of utility implies the unwillingness to accept a fair lottery, we can use a result from statistics known as Jensen's inequality. If $U(\cdot)$ is some concave function, and $\widetilde{x}$ is a random variable, then Jensen's inequality says that

$$E[U(\tilde{x})] < U(E[\tilde{x}]) \tag{1.15}$$

Therefore, substituting $\tilde{x} = W + \widetilde{\varepsilon}$, with $E[\widetilde{\varepsilon}] = 0$, we have

$$E\left[U(W + \widetilde{\varepsilon})\right] < U\left(E\left[W + \widetilde{\varepsilon}\right]\right) = U(W) \tag{1.16}$$

which is the desired result.

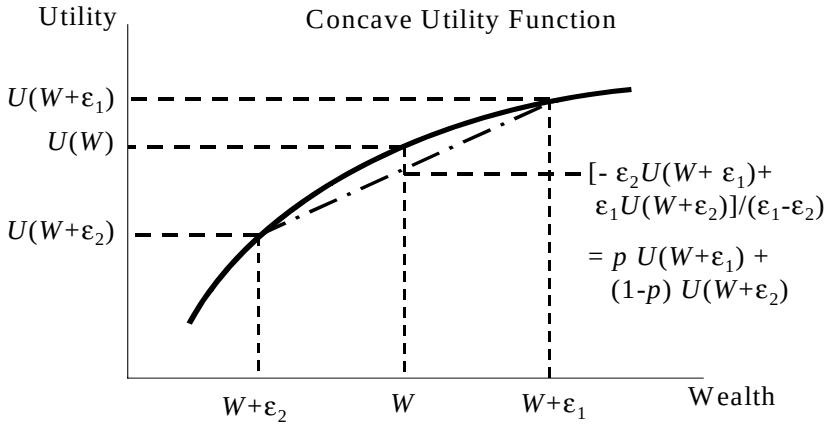



Figure 1.1: Fair Lotteries Lower Utility

We have defined risk aversion in terms of the individual's utility function.[12] Let us now consider how this aversion to risk can be quantified. This is done by defining a *risk premium*, the amount that an individual is willing to pay to avoid a risk.

Let $\pi$ denote the individual's risk premium for a particular lottery, $\widetilde{\varepsilon}$. It can be likened to the maximum insurance payment an individual would pay to avoid a particular risk. John W. Pratt (Pratt 1964) defined the risk premium for lottery (asset) $\widetilde{\varepsilon}$ as

$$U(W-\pi) = E\left[U(W+\widetilde{\varepsilon})\right] \tag{1.17}$$

[12] Based on the same analysis, it is straightforward to show that if an individual strictly preferred a fair lottery, his utility function must be convex in wealth. Such an individual is said to be risk-loving. Similarly, an individual that is indifferent between accepting or refusing a fair lottery is said to be risk-neutral and must have utility that is a linear function of wealth.

$W - \pi$ is defined as the *certainty equivalent* level of wealth associated with the lottery, $\widetilde{\varepsilon}$. Since utility is an increasing, concave function of wealth, Jensen's inequality ensures that $\pi$ must be positive when $\widetilde{\varepsilon}$ is fair, that is, the individual would accept a level of wealth lower than her expected level of wealth following the lottery, $E[W + \widetilde{\varepsilon}]$, if the lottery could be avoided.

To analyze this Pratt (1964) risk premium, we continue to assume the individual is an expected utility maximizer and that $\widetilde{\varepsilon}$ is a fair lottery, that is, its expected value equals zero. Further, let us consider the case of $\widetilde{\varepsilon}$ being "small," so that we can study its effects by taking a Taylor series approximation of equation (1.17) around the point $\widetilde{\varepsilon} = 0$ and $\pi = 0$.[13] Expanding the left hand side of (1.17) around $\pi = 0$ gives

$$U(W - \pi) \cong U(W) - \pi U'(W) \tag{1.18}$$

and expanding the right hand side of (1.17) around $\widetilde{\varepsilon} = 0$ (and taking a three term expansion since $E[\widetilde{\varepsilon}] = 0$ implies that a third term is necessary for a limiting approximation) gives

$$E[U(W + \widetilde{\varepsilon})] \cong E\left[U(W) + \widetilde{\varepsilon}U'(W) + \tfrac{1}{2}\widetilde{\varepsilon}^2 U''(W)\right] \tag{1.19}$$

$$= U(W) + \tfrac{1}{2}\sigma^2 U''(W)$$

where $\sigma^2 \equiv E\left[\widetilde{\varepsilon}^2\right]$ is the lottery's variance. Equating the results in (1.18) and (1.19), we have

$$\pi = -\tfrac{1}{2}\sigma^2 \frac{U''(W)}{U'(W)} \equiv \tfrac{1}{2}\sigma^2 R(W) \tag{1.20}$$

[13] By describing the random variable $\widetilde{\varepsilon}$ as "small" we mean that its probability density is concentrated around its mean of 0.

where $R(W) \equiv -U''(W)/U'(W)$ is the Pratt (1964) - Arrow (1971) measure of absolute risk aversion. Note that the risk premium, $\pi$, depends on the uncertainty of the risky asset, $\sigma^2$, and on the individual's coefficient of absolute risk aversion. Since $\sigma^2$ and $U'(W)$ are both greater than zero, concavity of the utility function ensures that $\pi$ must be positive.

From (1.20) we see that the concavity of the utility function, $U''(W)$, is insufficient to quantify the risk premium an individual is willing to pay, even though it is necessary and sufficient to indicate whether the individual is risk-averse. In order to determine the risk premium, we also need the first derivative, $U'(W)$, which tells us the marginal utility of wealth. An individual may be very risk averse ($-U''(W)$ is large), but he may be unwilling to pay a large risk premium if he is poor since his marginal utility is high ($U'(W)$ is large).

To illustrate this point, consider the following *negative exponential* utility function:

$$U(W) = -e^{-bW}, b > 0 \tag{1.21}$$

Note that $U'(W) = be^{-bW} > 0$ and $U''(W) = -b^2e^{-bW} < 0$. Consider the behavior of a very wealthy individual, that is, one whose wealth approaches infinity:

$$\lim_{W\to\infty} U'(W) = \lim_{W\to\infty} U''(W) = 0 \tag{1.22}$$

As $W \to \infty$, the utility function is a flat line. Concavity disappears, which might imply that this very rich individual would be willing to pay very little for insurance against a random event, $\widetilde{\varepsilon}$, certainly less than a poor person with the same utility function. However, this is not true because the marginal utility of wealth is also very small. This neutralizes the effect of smaller concavity.

Indeed:

$$R(W) = \frac{b^2 e^{-bW}}{be^{-bW}} = b \tag{1.23}$$

which is a constant. Thus, we can see why this utility function is sometimes referred to as a *constant absolute risk aversion* utility function.

If we want to assume that absolute risk aversion is declining in wealth, a necessary, though not sufficient, condition for this is that the utility function have a positive third derivative, since

$$\frac{\partial R(W)}{\partial W} = -\frac{U'''(W)U'(W) - [U''(W)]^2}{[U'(W)]^2} \tag{1.24}$$

Also, it can be shown that the coefficient of risk aversion contains all relevant information about the individual's risk preferences. To see this, note that

$$R(W) = -\frac{U''(W)}{U'(W)} = -\frac{\partial \left(\ln \left[U'(W)\right]\right)}{\partial W} \tag{1.25}$$

Integrating both sides of (1.25), we have

$$-\int R(W)dW = \ln[U'(W)] + c_1 \tag{1.26}$$

where $c_1$ is an arbitrary constant. Taking the exponential function of (1.26), one obtains

$$e^{-\int R(W)dW} = U'(W)e^{c_1} \tag{1.27}$$

Integrating once again gives

$$\int e^{-\int R(W)dW} dW = e^{c_1} U(W) + c_2 \tag{1.28}$$

where $c_2$ is another arbitrary constant. Because expected utility functions are unique up to a linear transformation, $e^{c_1}U(W) + c_1$ reflects the same risk preferences as $U(W)$. Hence, this shows one can recover the risk-preferences of $U(W)$ from the function $R(W)$.

*Relative* risk aversion is another frequently used measure of risk aversion and is defined simply as

$$R_r(W) = WR(W) \tag{1.29}$$

In many applications in financial economics, an individual is assumed to have relative risk aversion that is constant for different levels of wealth. Note that this assumption implies that the individual's absolute risk aversion, $R(W)$, declines in direct proportion to increases in his wealth. While later chapters will discuss the widely varied empirical evidence on the size of individuals' relative risk aversions, one recent study based on individuals' answers to survey questions finds a median relative risk aversion of approximately 7.[14]

Let us now examine the coefficients of risk aversion for some utility functions that are frequently used in models of portfolio choice and asset pricing. *Power* utility can be written as

$$U(W) = \tfrac{1}{\gamma}W^{\gamma}, \gamma < 1 \tag{1.30}$$

[14] The mean estimate was lower, indicating a skewed distribution. Robert Barsky, Thomas Juster, Miles Kimball, and Matthew Shapiro (Barsky, Juster, Kimball, and Shapiro 1997) computed these estimates of relative risk aversion from a survey that asked a series of questions regarding whether the respondent would switch to a new job that had a 50-50 chance of doubling their lifetime income or decreasing their lifetime income by a proportion $\lambda$. By varying $\lambda$ in the questions, they estimated the point where an individual would be indifferent between keeping their current job or switching. Essentially, they attempted to find $\lambda^*$ such that $\frac{1}{2}U(2W) + \frac{1}{2}U(\lambda^* W) = U(W)$. Assuming utility displays constant relative risk aversion of the form $U(W) = W^{\gamma}/\gamma$, then the coefficient of relative risk aversion, $1-\gamma$ satisfies $2^{\gamma} + \lambda^{*\gamma} = 2$. The authors warn that their estimates of risk aversion may be biased upward if individuals attach non-pecuniary benefits to maintaining their current occupation. Interestingly, they confirmed that estimates of relative risk aversion tended to be lower for individuals who smoked, drank, were uninsured, held riskier jobs, and invested in riskier assets.

implying that $R(W) = -\frac{(\gamma-1)W^{\gamma-2}}{W^{\gamma-1}} = \frac{(1-\gamma)}{W}$ and, therefore, $R_r(W) = 1 - \gamma$. Hence, this form of utility is also known as *constant relative risk aversion*. *Logarithmic* utility is a limiting case of power utility. To see this, write the power utility function as $\frac{1}{\gamma}W^{\gamma} - \frac{1}{\gamma} = \frac{W^{\gamma}-1}{\gamma}$.[15] Next take the limit of this utility function as $\gamma \to 0$. Note that the numerator and denominator both go to zero, so that the limit is not obvious. However, we can re-write the numerator in terms of an exponential and natural log function and apply L'Hôpital's rule to obtain:

$$\lim_{\gamma\to 0}\frac{W^{\gamma}-1}{\gamma} = \lim_{\gamma\to 0}\frac{e^{\gamma\ln(W)}-1}{\gamma} = \lim_{\gamma\to 0}\frac{\ln(W)W^{\gamma}}{1} = \ln(W) \tag{1.31}$$

Thus, logarithmic utility is equivalent to power utility with $\gamma = 0$, or a coefficient of relative risk aversion of unity:

$R(W) = -\frac{W^{-2}}{W^{-1}} = \frac{1}{W}$ and $R_r(W) = 1$.

*Quadratic* utility takes the form

$$U(W) = W - \tfrac{b}{2}W^2, b > 0 \tag{1.32}$$

Note that the marginal utility of wealth is $U'(W) = 1 - bW$ and is positive only when $b < \frac{1}{W}$. Thus, this utility function makes sense (in that more wealth is preferred to less) only when $W < \frac{1}{b}$. The point of maximum utility, $\frac{1}{b}$, is known as the "bliss point." We have $R(W) = \frac{b}{1-bW}$ and $R_r(W) = \frac{bW}{1-bW}$.

*Hyperbolic absolute risk aversion* (HARA) utility is a generalization of all of the aforementioned utility functions. It can be written as

$$U(W) = \frac{1-\gamma}{\gamma}\left(\frac{\alpha W}{1-\gamma} + \beta\right)^{\gamma} \tag{1.33}$$

[15] Recall that we can do this because utility functions are unique up to a linear transformation.

subject to the restrictions $\gamma \neq 1$, $\alpha > 0$, $\frac{\alpha W}{1-\gamma} + \beta > 0$, and $\beta = 1$ if $\gamma = -\infty$. Thus, $R(W) = \left(\frac{W}{1-\gamma} + \frac{\beta}{\alpha}\right)^{-1}$. Since $R(W)$ must be $> 0$, it implies $\beta > 0$ when $\gamma > 1$. $R_r(W) = W\left(\frac{W}{1-\gamma} + \frac{\beta}{\alpha}\right)^{-1}$. HARA utility nests constant absolute risk aversion ($\gamma = -\infty$, $\beta = 1$), constant relative risk aversion ($\gamma < 1$, $\beta = 0$), and quadratic ($\gamma = 2$) utility functions. Thus, depending on the parameters, it is able to display constant absolute risk aversion or relative risk aversion that is increasing, decreasing, or constant. We will re-visit HARA utility in future chapters as it can be an analytically convenient assumption for utility when deriving an individual's intertemporal consumption and portfolio choices.

Pratt's definition of a risk premium in (1.17) is commonly used in the insurance literature because it can be interpreted as the payment that an individual is willing to make to insure against a particular risk. However, in the field of financial economics, a somewhat different definition is often employed. Financial economists seek to understand how the risk of an asset's payoff determines the asset's rate of return. In this context, an asset's risk premium is defined as its expected rate of return in excess of the risk-free rate of return. This alternative concept of a risk premium was used by Kenneth Arrow (Arrow 1971) who independently derived a coefficient of risk aversion that is identical to Pratt's measure. Let us now outline Arrow's approach. Suppose that an asset (lottery), $\widetilde{\varepsilon}$, has the following payoffs and probabilities (this could be generalized to other types of fair payoffs):

$$\widetilde{\varepsilon} = \begin{cases} +\epsilon \text{ with probability } \frac{1}{2} \\ -\epsilon \text{ with probability } \frac{1}{2} \end{cases} \tag{1.34}$$

where $\epsilon \geq 0$. Note that, as before, $E[\widetilde{\varepsilon}] = 0$. Now consider the following question. By how much should we change the expected value (return) of the asset, by changing the probability of winning, in order to make the individual

indifferent between taking and not taking the risk? If $p$ is the probability of winning, we can define the risk premium as

$$\theta = prob\left(\widetilde{\varepsilon} = +\epsilon\right) - prob\left(\widetilde{\varepsilon} = -\epsilon\right) = p - (1-p) = 2p - 1 \tag{1.35}$$

Therefore, from (1.35) we have

$$\begin{array}{c} prob\left(\widetilde{\varepsilon} = +\epsilon\right) \equiv p = \frac{1}{2}(1+\theta) \\ prob\left(\widetilde{\varepsilon} = -\epsilon\right) \equiv 1 - p = \frac{1}{2}(1-\theta) \end{array} \tag{1.36}$$

These new probabilities of winning and losing are equal to the old probabilities, $\frac{1}{2}$, plus half of the increment, $\theta$. Thus, the premium, $\theta$, that makes the individual indifferent between accepting and refusing the asset is

$$U(W) = \frac{1}{2}(1+\theta)U(W+\epsilon) + \frac{1}{2}(1-\theta)U(W-\epsilon) \tag{1.37}$$

Taking a Taylor series approximation around $\epsilon = 0$, gives

$$\begin{array}{rcl} U(W) & = & \frac{1}{2}(1+\theta)\left[U(W) + \epsilon U'(W) + \frac{1}{2}\epsilon^2 U''(W)\right] \\ & & + \frac{1}{2}(1-\theta)\left[U(W) - \epsilon U'(W) + \frac{1}{2}\epsilon^2 U''(W)\right] \\ & = & U(W) + \epsilon\theta U'(W) + \frac{1}{2}\epsilon^2 U''(W) \end{array} \tag{1.38}$$

Re-arranging (1.38) implies

$$\theta = \tfrac{1}{2}\epsilon R(W) \tag{1.39}$$

which, as before, is a function of the coefficient of absolute risk aversion. Note that the Arrow premium, $\theta$, is in terms of a probability, while the Pratt measure, $\pi$, is in units of a monetary payment. If we multiply $\theta$ by the monetary payment

received, $\epsilon$, then equation (1.39) becomes

$$\epsilon\theta = \tfrac{1}{2}\epsilon^2 R(W) \tag{1.40}$$

Since $\epsilon^2$ is the variance of the random payoff, $\widetilde{\varepsilon}$, equation (1.40) shows that the Pratt and Arrow measures of risk premia are equivalent. Both were obtained as a linearization of the true function around $\widetilde{\varepsilon} = 0$.

The results of this section showed how risk aversion depends on the shape of an individual's utility function. Moreover, it demonstrated that a risk premium, equal to either the payment an individual would make to avoid a risk or the individual's required excess rate of return on a risky asset, is proportional to the individual's Pratt-Arrow coefficient of absolute risk aversion.

## 1.3 Risk Aversion and Portfolio Choice

Having developed the concepts of risk aversion and risk premiums, we now consider the relation between risk aversion and an individual's portfolio choice in a single period context. While the portfolio choice problem that we analyze is very simple, many of its insights extend to the more complex environments that will be covered in later chapters of this book. We shall demonstrate that absolute and relative risk aversion play important roles in determining how portfolio choices vary with an individual's level of wealth. Moreover, we show that when given a choice between a risk-free asset and a risky asset, a risk-averse individual always chooses at least some positive investment in the risky asset if it pays a positive risk premium.

The model's assumptions are as follows. Assume there is a riskless security that pays a rate of return equal to $r_f$. In addition, for simplicity suppose there is just one risky security that pays a stochastic rate of return equal to $\widetilde{r}$. Also,

let $W_0$ be the individual's initial wealth, and let $A$ be the dollar amount that the individual invests in the risky asset at the beginning of the period. Thus, $W_0 - A$ is the initial investment in the riskless security. Denoting the individual's end-of-period wealth as $\tilde{W}$, it satisfies:

$$\begin{aligned} \tilde{W} &= (W_0 - A)(1 + r_f) + A(1 + \tilde{r}) \\ &= W_0(1 + r_f) + A(\tilde{r} - r_f) \end{aligned} \tag{1.41}$$

Note that in the second line of equation (1.41), the first term is the individual's return on wealth when the entire portfolio is invested in the risk-free asset, while the second term is the difference in return gained by investing $A$ dollars in the risky asset.

We assume that the individual cares only about consumption at the end of this single period. Therefore, maximizing end-of-period consumption is equivalent to maximizing end-of-period wealth. Assuming that the individual is a von Neumann-Morgenstern expected utility maximizer, she chooses her portfolio by maximizing the expected utility of end-of-period wealth:

$$\max_A E[U(\tilde{W})] = \max_A E\left[U\left(W_0(1 + r_f) + A(\tilde{r} - r_f)\right)\right] \tag{1.42}$$

The solution to the individual's problem in (1.42) must satisfy the following first order condition with respect to $A$:

$$E\left[U'\left(\tilde{W}\right)(\tilde{r} - r_f)\right] = 0 \tag{1.43}$$

This condition determines the amount, $A$, that the individual invests in the

risky asset.[16] Consider the special case in which the expected rate of return on the risky asset equals the risk-free rate. In that case $A = 0$ satisfies the first order condition. To see this, note that when $A = 0$, then $\tilde{W} = W_0(1 + r_f)$ and, therefore, $U'\left(\tilde{W}\right) = U'(W_0(1 + r_f))$ are non-stochastic. Hence, $E\left[U'\left(\tilde{W}\right)(\tilde{r} - r_f)\right] = U'(W_0(1 + r_f))E[\tilde{r} - r_f] = 0$. This result is reminiscent of our earlier finding that a risk-averse individual would not choose to accept a fair lottery. Here, the fair lottery is interpreted as a risky asset that has an expected rate of return just equal to the risk-free rate.

Next, consider the case in which $E[\tilde{r}] - r_f > 0$. Clearly, $A = 0$ would not satisfy the first order condition because $E\left[U'\left(\tilde{W}\right)(\tilde{r} - r_f)\right] = U'(W_0(1 + r_f))E[\tilde{r} - r_f] > 0$ when $A = 0$. Rather, when $E[\tilde{r}] - r_f > 0$ condition (1.43) is satisfied only when $A > 0$. To see this, let $r^h$ denote a realization of $\tilde{r}$ such that it exceeds $r_f$, and let $W^h$ be the corresponding level of $\tilde{W}$. Also let $r^l$ denote a realization of $\tilde{r}$ such that it is lower than $r_f$, and let $W^l$ be the corresponding level of $\tilde{W}$. Obviously, $U'(W^h)(r^h - r_f) > 0$ and $U'(W^l)(r^l - r_f) < 0$. For $U'\left(\tilde{W}\right)(\tilde{r} - r_f)$ to average to zero for all realizations of $\tilde{r}$, it must be the case that $W^h > W^l$ so that $U'\left(W^h\right) < U'\left(W^l\right)$ due to the concavity of the utility function. This is because since $E[\tilde{r}] - r_f > 0$, the average realization of $r^h$ is farther above $r_f$ than the average realization of $r^l$ is below $r_f$. Therefore, to make $U'\left(\tilde{W}\right)(\tilde{r} - r_f)$ average to zero, the positive $(r^h - r_f)$ terms need to be given weights, $U'\left(W^h\right)$, that are smaller than the weights, $U'(W^l)$, that multiply the negative $(r^l - r_f)$ realizations. This can occur only if $A > 0$ so that $W^h > W^l$. The implication is that an individual will always hold at least some positive amount of the risky asset if its expected rate of return exceeds the risk-free rate.[17]

[16] The second order condition for a maximum, $E\left[U''\left(\tilde{W}\right)(\tilde{r} - r_f)^2\right] \leq 0$, is satisfied because $U''\left(\tilde{W}\right) \leq 0$ due to the assumed concavity of the utility function.

[17] Related to this is the notion that a risk-averse expected utility maximizer should accept a small lottery with a positive expected return. In other words, such an individual should be close to risk-neutral for small-scale bets. However, Matthew Rabin and Richard Thaler (Rabin and Thaler 2001) claim that individuals frequently reject lotteries (gambles) that are

Now, we can go further and explore the relationship between $A$ and the individual's initial wealth, $W_0$. Using the envelope theorem, we can differentiate the first order condition to obtain[18]

$$E\left[U''(\tilde{W})(\tilde{r}-r_f)(1+r_f)\right]dW_0 + E\left[U''(\tilde{W})(\tilde{r}-r_f)^2\right]dA = 0 \tag{1.44}$$

or

$$\frac{dA}{dW_0} = \frac{(1+r_f)E\left[U''(\tilde{W})(\tilde{r}-r_f)\right]}{-E\left[U''(\tilde{W})(\tilde{r}-r_f)^2\right]} \tag{1.45}$$

The denominator of (1.45) is positive because concavity of the utility function ensures that $U''(\tilde{W})$ is negative. Therefore, the sign of the expression depends on the numerator, which can be of either sign because realizations of $(\tilde{r}-r_f)$ can turn out to be both positive and negative.

To characterize situations in which the sign of (1.45) can be determined, let us first consider the case where the individual has absolute risk aversion that is decreasing in wealth. As before, let $r^h$ denote a realization of $\tilde{r}$ such that it exceeds $r_f$, and let $W^h$ be the corresponding level of $\tilde{W}$. Then for $A \geqslant 0$, we have $W^h \geqslant W_0(1+r_f)$. If absolute risk aversion is decreasing in wealth, this

modest in size yet have positive expected returns. From this they argue that concave expected utility is not a plausible model for predicting an individual's choice of small-scale risks.

[18] The envelope theorem is used to analyze how the maximized value of the objective function and the control variable change when one of the model's parameters changes. In our context, define $f(A, W_0) \equiv E\left[U\left(\widetilde{W}\right)\right]$ so that $v(W_0) = \max_A f(A, W_0)$ is the maximized value of the objective function when the control variable, $A$, is optimally chosen. Also define $A(W_0)$ as the value of $A$ that maximizes $f$ for a given value of $W_0$. Then applying the chain rule, we have $\frac{dv(W_0)}{dW_0} = \frac{\partial f(A,W_0)}{\partial A}\frac{dA(W_0)}{dW_0} + \frac{\partial f(A(W_0),W_0)}{\partial W_0}$. But since $\frac{\partial f(A,W_0)}{\partial A} = 0$, from the first order condition, this simplifies to just $\frac{dv(W_0)}{dW_0} = \frac{\partial f(A(W_0),W_0)}{\partial W_0}$. Again applying the chain rule to the first order condition, one obtains $\frac{\partial(\partial f(A(W_0),W_0)/\partial A)}{\partial W_0} = 0 = \frac{\partial^2 f(A(W_0),W_0)}{\partial A^2}\frac{dA(W_0)}{dW_0} + \frac{\partial^2 f(A(W_0),W_0)}{\partial A \partial W_0}$. Re-arranging gives us $\frac{dA(W_0)}{dW_0} = -\frac{\partial^2 f(A(W_0),W_0)}{\partial A \partial W_0} / \frac{\partial^2 f(A(W_0),W_0)}{\partial A^2}$, which is equation (1.45).

implies

$$R\left(W^h\right) \leqslant R\left(W_0(1+r_f)\right) \tag{1.46}$$

where, as before, $R(W) = -U''(W)/U'(W)$. Multiplying both terms of (1.46) by $-U'(W^h)(r^h - r_f)$, which is a negative quantity, the inequality sign changes:

$$U''(W^h)(r^h - r_f) \geqslant -U'(W^h)(r^h - r_f)R\left(W_0(1+r_f)\right) \tag{1.47}$$

Next, we again let $r^l$ denote a realization of $\tilde{r}$ that is lower than $r_f$ and define $W^l$ to be the corresponding level of $\tilde{W}$. Then for $A \geqslant 0$, we have $W^l \leqslant W_0(1+r_f)$. If absolute risk aversion is decreasing in wealth, this implies

$$R(W^l) \geqslant R\left(W_0(1+r_f)\right) \tag{1.48}$$

Multiplying (1.48) by $-U'(W^l)(r^l - r_f)$, which is positive, so that the sign of (1.48) remains the same, we obtain

$$U''(W^l)(r^l - r_f) \geqslant -U'(W^l)(r^l - r_f)R\left(W_0(1+r_f)\right) \tag{1.49}$$

Notice that inequalities (1.47) and (1.49) are of the same form. The inequality holds whether the realization is $\tilde{r} = r^h$ or $\tilde{r} = r^l$. Therefore, if we take expectations over all realizations, where $\tilde{r}$ can be either higher than or lower than $r_f$, we obtain

$$E\left[U''(\tilde{W})(\tilde{r} - r_f)\right] \geqslant -E\left[U'(\tilde{W})(\tilde{r} - r_f)\right] R\left(W_0(1+r_f)\right) \tag{1.50}$$

Since the first term on the right-hand-side is just the first order condition, inequality (1.50) reduces to

$$E\left[U''(\tilde{W})(\tilde{r}-r_f)\right] \gtrless 0 \tag{1.51}$$

Thus, the first conclusion that can be drawn is that declining absolute risk aversion implies $dA/dW_0 > 0$, that is, the individual invests an increasing amount of wealth in the risky asset for larger amounts of initial wealth. For two individuals with the same utility function but different initial wealths, the more wealthy one invests a greater dollar amount in the risky asset if utility is characterized by decreasing absolute risk aversion. While not shown here, the opposite is true, namely, that the more wealthy individual invests a smaller dollar amount in the risky asset if utility is characterized by increasing absolute risk aversion.

Thus far, we have not said anything about the *proportion* of initial wealth invested in the risky asset. To analyze this issue, we need the concept of relative risk aversion. Define

$$\eta \equiv \frac{dA}{dW_0}\frac{W_0}{A} \tag{1.52}$$

which is the elasticity measuring the proportional increase in the risky asset for an increase in initial wealth. Adding $1-\frac{A}{A}$ to the right hand side of (1.52) gives

$$\eta = 1 + \frac{(dA/dW_0)W_0 - A}{A} \tag{1.53}$$

Substituting the expression $dA/dW_0$ from equation (1.45), we have

$$\eta = 1 + \frac{W_0(1+r_f)E\left[U''(\tilde{W})(\tilde{r}-r_f)\right] + AE\left[U''(\tilde{W})(\tilde{r}-r_f)^2\right]}{-AE\left[U''(\tilde{W})(\tilde{r}-r_f)^2\right]} \tag{1.54}$$

Collecting terms in $U''(\tilde{W})(\tilde{r}-r_f)$, this can be re-written as

$$\begin{aligned}\eta &= 1+\frac{E\left[U''(\tilde{W})(\tilde{r}-r_f)\{W_0(1+r_f)+A(\tilde{r}-r_f)\}\right]}{-AE\left[U''(\tilde{W})(\tilde{r}-r_f)^2\right]} \qquad (1.55)\\ &= 1+\frac{E\left[U''(\tilde{W})(\tilde{r}-r_f)\tilde{W}\right]}{-AE\left[U''(\tilde{W})(\tilde{r}-r_f)^2\right]}\end{aligned}$$

The denominator is always positive. Therefore, we see that the elasticity, $\eta$, is greater than one, so that the individual invests proportionally more in the risky asset with an increase in wealth, if $E\left[U''(\tilde{W})(\tilde{r}-r_f)\tilde{W}\right] \geqslant 0$. Can we relate this to the individual's risk aversion? The answer is yes and the derivation is almost exactly the same as that just given.

Consider the case where the individual has *relative* risk aversion that is decreasing in wealth. Let $r^h$ denote a realization of $\tilde{r}$ such that it exceeds $r_f$, and let $W^h$ be the corresponding level of $\tilde{W}$. Then for $A \geqslant 0$, we have $W^h \geqslant W_0(1+r_f)$. If relative risk aversion, $R_r(W) \equiv WR(W)$, is decreasing in wealth, this implies

$$W^h R(W^h) \leqslant W_0(1+r_f)R\left(W_0(1+r_f)\right) \qquad (1.56)$$

Multiplying both terms of (1.56) by $-U'(W^h)(r^h-r_f)$, which is a negative quantity, the inequality sign changes:

$$W^h U''(W^h)(r^h-r_f) \geqslant -U'(W^h)(r^h-r_f)W_0(1+r_f)R\left(W_0(1+r_f)\right) \qquad (1.57)$$

Next, let $r^l$ denote a realization of $\tilde{r}$ such that it is lower than $r_f$, and let $W^l$ be the corresponding level of $\tilde{W}$. Then for $A \geqslant 0$, we have $W^l \leqslant W_0(1+r_f)$. If relative risk aversion is decreasing in wealth, this implies

$$W^l R(W^l) \geqslant W_0(1+r_f)R\left(W_0(1+r_f)\right) \tag{1.58}$$

Multiplying (1.58) by $-U'(W^l)(r^l - r_f)$, which is positive, so that the sign of (1.58) remains the same, we obtain

$$W^l U''(W^l)(r^l - r_f) \geqslant -U'(W^l)(r^l - r_f)W_0(1+r_f)R\left(W_0(1+r_f)\right) \tag{1.59}$$

Notice that inequalities (1.57) and (1.59) are of the same form. The inequality holds whether the realization is $\tilde{r} = r^h$ or $\tilde{r} = r^l$. Therefore, if we take expectations over all realizations, where $\tilde{r}$ can be either higher than or lower than $r_f$, we obtain

$$E\left[\tilde{W}U''(\tilde{W})(\tilde{r} - r_f)\right] \geqslant -E\left[U'(\tilde{W})(\tilde{r} - r_f)\right] W_0(1+r_f)R(W_0(1+r_f)) \tag{1.60}$$

Since the first term on the right-hand-side is just the first order condition, inequality (1.60) reduces to

$$E\left[\tilde{W}U''(\tilde{W})(\tilde{r} - r_f)\right] \geqslant 0 \tag{1.61}$$

Thus, we see that an individual with decreasing relative risk aversion has $\eta > 1$ and invests proportionally more in the risky asset as wealth increases. The opposite is true for increasing relative risk aversion: $\eta < 1$ so that this individual invests proportionally less in the risky asset as wealth increases. The following table provides another way of writing this section's main results.

| **Risk Aversion** | **Investment Behavior** |
|---|---|
| Decreasing Absolute | $\frac{\partial A}{\partial W_0} > 0$ |
| Constant Absolute | $\frac{\partial A}{\partial W_0} = 0$ |
| Increasing Absolute | $\frac{\partial A}{\partial W_0} < 0$ |
| Decreasing Relative | $\frac{\partial A}{\partial W_0} > \frac{A}{W_0}$ |
| Constant Relative | $\frac{\partial A}{\partial W_0} = \frac{A}{W_0}$ |
| Increasing Relative | $\frac{\partial A}{\partial W_0} < \frac{A}{W_0}$ |

A point worth emphasizing is that absolute risk aversion indicates how the investor's dollar amount in the risky asset changes with changes in initial wealth while relative risk aversion indicates how the investor's portfolio proportion (or portfolio weight) in the risky asset, $A/W_0$, changes with changes in initial wealth.

## 1.4 Summary

This chapter is a first step toward understanding how an individual's preferences toward risk affects his portfolio behavior. It was shown that if an individual's risk preferences satisfied specific plausible conditions, then her behavior could be represented by a von Neumann-Morgenstern expected utility function. In turn, the shape of the individual's utility function determines a measure of risk aversion that is linked to two concepts of a risk premium. The first one is the monetary payment that the individual is willing to pay to avoid a risk, an example being a premium paid to insure against a property/casualty loss. The second is the rate of return in excess of a riskless rate that the individual requires to hold a risky asset, which is the common definition of a security risk premium used in the finance literature. Finally, it was shown how an individual's absolute and relative risk aversion affects his choice between a risky and risk-free asset. In

particular, individuals with decreasing (increasing) relative risk aversion invest proportionally more (less) in the risky asset as their wealth increases. Though based on a simple single-period, two asset portfolio choice model, this insight generalizes to the more complex portfolio choice problems that will be studied in later chapters.

## 1.5 Exercises

1. Suppose there are two lotteries $P = \{p_1, ..., p_n\}$ and $P^* = \{p_1^*, ..., p_n^*\}$. Let $V(p_1, ..., p_n) = \sum_{i=1}^{n} p_i U_i$ be an individual's expected utility function defined over these lotteries. Let $W(p_1, ..., p_n) = \sum_{i=1}^{n} p_i Q_i$ where $Q_i = a + bU_i$ and $a$ and $b$ are constants. If $P^* \succ P$, so that $V(p_1^*, ..., p_n^*) > V(p_1, ..., p_n)$, must it be the case that $W(p_1^*, ..., p_n^*) > W(p_1, ..., p_n)$? In other words, is $W$ also a valid expected utility function for the individual? Are there any restrictions needed on $a$ and $b$ for this to be the case?

2. (Allais Paradox) Asset A pays \$1,500 with certainty, while asset B pays \$2,000 with probability 0.8 or \$100 with probability 0.2. If offered the choice between asset A or B, a particular individual would choose asset A. Suppose, instead, the individual is offered the choice between asset C and asset D. Asset C pays \$1,500 with probability 0.25 or \$100 with probability 0.75 while asset D pays \$2,000 with probability 0.2 or \$100 with probability 0.8. If asset D is chosen, show that the individual's preferences violate the independence axiom.

3. Verify that the HARA utility function in 1.33 becomes the constant absolute risk aversion utility function when $\beta = 1$ and $\gamma = -\infty$. Hint: recall that $e^a = \lim_{x \longrightarrow \infty} \left(1 + \frac{a}{x}\right)^x$.

4. Consider the individual's portfolio choice problem given in 1.42. As-